Some Maths on Interest Rates

Tong Xiao

Imperial College London

February 26, 2025

Contents

1 Basics

1.1 Risk Neutral Measure

Definition 1.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

A **measure** is a function $\mu : \mathcal{F} \to [0, \infty]$ such that:

$$\mu(\emptyset) = 0$$
$$\mu(A) \geq 0 \; \forall A \in \mathcal{F}$$
$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$

for any disjoint sequence of sets $A_i \in \mathcal{F}$.

A **probability measure** is a measure $\mathbb{P}$ such that $\mathbb{P}(\Omega) = 1$.

Definition 1.2. Given measure space $(X, \mathcal{F}, \mu)$, and let q be a σ-finite measure on $(X, \mathcal{F})$ We say that q is **absolutely continuous** with respect to μ, denoted as $q \ll \mu$, if $\mu(A) = 0$ implies $q(A) = 0$. The two are **equivalent martingale measure** if $q \ll \mu$ and $\mu \ll q$, or

$$q(A) = 0 \iff \mu(A) = 0$$

Definition 1.3. A **risk-neutral measure** is a probability measure $\mathbb{Q}$ such that for any tradble asset S_t, its discounted price process $S_t^* = S_t/B_t$ is a martingale under $\mathbb{Q}$:

For all $T > t$:

$$E_{\mathbb{Q}}[S_T/B_T|\mathcal{F}_t] = S_t/B_t$$

An easy corollary is that

$$E_{\mathbb{Q}}[S_t]/B_t = S_0/B_0 = S_0$$
$$E_{\mathbb{Q}}[S_t]/S_0 = B_t/B_0 = e^{rt}$$

This is also equivalent to saying that under Q, the drift for any S_t is rS_t:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q$$

Definition 1.4. A basic portfolio π with weight vector $\pi_t = (\pi_{t,0}, \pi_{t,1})$ has price process V_t:

$$V_t = \pi_{t,0} B_t + \pi_{t,1} S_t$$

A portfolio π with price process V_t is **self-financing** if:

$$dV_t = \pi_{t,0}\, dB_t + \pi_{t,1}\, dS_t$$

In discrete time, we can write

$$V_t = \pi_{t-1,0} B_t + \pi_{t-1,1} S_t$$

i.e. A portfolio's only change in value is due to change in price of B_t and S_t, so not external cash flow

Definition 1.5. A portfolio π is **arbitrage** if it is **self-financing**, adapted to the filtration $\mathcal{F}_t$, and:

$$V_0(\pi) = 0$$
$$P(V_T(\pi) \geq 0) = 1$$
$$P(V_T(\pi) > 0) > 0$$

Theorem 1.1. *Any price process V_t is* ***arbitrage-free*** *iff there is an EMM $\mathbb{Q}$.*
In this case the discounted $\frac{V_t}{B_t}$ is a martingale under $\mathbb{Q}$.

Proof. By self-financing, we have

$$\begin{aligned} V_t =& \pi_{t-1,0} B_t + \pi_{t-1,1} S_t \\ \frac{V_t}{B_t} =& \pi_{t-1,0} + \pi_{t-1,1} \frac{S_t}{B_t} \end{aligned}$$

Since $\frac{S_t}{B_t}$ is a martingale under $\mathbb{Q}$, we have that at time $t-1$

$$\begin{aligned} E_Q\left[\frac{V_t}{B_t} | \mathcal{F}_{t-1}\right] =& \pi_{t-1,0} + \pi_{t-1,1} E_Q\left[\frac{S_t}{B_t} | \mathcal{F}_{t-1}\right] \\ =& \pi_{t-1,0} + \pi_{t-1,1} \frac{S_{t-1}}{B_{t-1}} \\ =& \frac{V_{t-1}}{B_{t-1}} \end{aligned}$$

So $\frac{V_t}{B_t}$ is a martingale under $\mathbb{Q}$.
We can repeatedly apply to show

$$E_Q\left[\frac{V_t}{B_t}\right] = \frac{V_0}{B_0} = 0$$

If π were an arbitrage, we would have $P(V_T(\pi) \geq 0) = 1 \implies Q(V_T(\pi) \geq 0) = 1$, This suggests that under Q, π is nonnegative but with expectation 0, so it has to be 0 a.s. But we also have $P(V_T(\pi) > 0) > 0 \implies Q(V_T(\pi) > 0) > 0$, hence a contradtion. □

1.2 To Find Q

We note $S_t^* = \frac{S_t}{B_t}$ is a martingale under $\mathbb{Q}$, with B_t being the numeraire.
To work out Q, we simply solve

$$E_{\mathbb{Q}}[S_T^*|\mathcal{F}_t] = S_t^*$$

Or equivalently,

$$E_{\mathbb{Q}}[S_T|\mathcal{F}_t] = S_t\frac{B_T}{B_t} = S_te^{r(T-t)}$$

1.2.1 Binomial Model

A simple example is the binomial model, where at t, there are two possible outcomes for $t+1$: S_tU and S_tD, with probabilities q^+ and q^- respectively.
So we can write

$$E_{\mathbb{Q}}[S_T|\mathcal{F}_t] = S_tUq^+ + S_tDq^-$$

We solve

$$\begin{aligned} S_te^{r(t+1-t)} =& S_tUq^+ + S_tDq^- \\ e^r =& Uq^+ + D(1-q^+) \end{aligned}$$

And we get $q^+ = \frac{e^r - D}{U-D}$, $q^- = \frac{U-e^r}{U-D}$
This is solvable iff $U > e^r > D$.

Example 1.2. *If a stock is at 100, can move to 90 or 120 in a year, with $r = 0$*
Real world probability of moving up or down is not important. Under Q, we have $q^+ = \frac{e^0 - 0.9}{1.2-0.9} = \frac{1}{3}$, $q^- = \frac{1-e^0}{1.2-0.9} = \frac{2}{3}$

Example 1.3. *We give a counterexample when $U > e^r > D$ does not hold.*
Consider a stock at 1, can move up to 3 or stay at 1.
The risk free rate is 0.

We have $q^+ = \frac{e^0-1}{3-1} = 0$, *so*

$$call = 0 \cdot 2 + 1 \cdot 0 = 0$$

Or we can buy infinite amount of calls on a stock with positive probability of going up.

1.2.2 Trinomial Model

Under the trinomial model, we have three possible outcomes for $t+1$:
S_tU, S_tM, S_tD with probabilities $q^+, (1-q^+-q^-), q^-$ respectively.
In this case we solve

$$\begin{aligned} S_tUq^+ + S_tM(1-q^+-q^-) + S_tDq^- &= S_te^r \\ Uq^+ + M(1-q^+-q^-) + Dq^- &= e^r \end{aligned}$$

This is solvable iff $U > e^r > D$.
In general, the trinomial model may have 0 or infinitely many solutions.

Theorem 1.4. *A market is* ***complete*** *if every payoff is attainable.*
In a complete market, there is a unique EMM $\mathbb{Q}$.

A binomial model is complete, and always has a unique solution Q, hence every payoff is attainable and has a unique price.
Whereas in a trinomial model, some payoffs have multiple EMMs, and hence multiple prices.

Example 1.5. *We hereby give an example of a trinomial model with multiple EMMs, but quoting unique price for an ITM put option.*
Consider

$$\begin{aligned} S_0 = 10, K = 25, r = \mu = 0.04, h = 0.1, T = 4 \\ U = e^{\mu+h} = e^{0.04+0.1} = e^{0.14}, \\ M = e^{\mu} = e^{0.04} = R = e^r, \\ D = e^{\mu-h} = e^{0.04-0.1} = e^{-0.06}, \end{aligned}$$

We have $U > R > D$, *so the model is solvable.*

We can solve the equation:

$$\frac{U}{R}q_+ + \frac{M}{R}(1-(q_+ + q_-)) + \frac{D}{R}q_- = 1$$
$$q_+\left(\frac{U-M}{R}\right) + q_-\left(\frac{D-M}{R}\right) = 1 - \frac{M}{R} = 1 - \frac{e^{0.04}}{e^{0.04}} = 0$$

This is a straight line with infinite solutions. Given any q_+, we have:

$$q_- = -\frac{U-M}{D-M}q_+ = -\frac{e^{0.14}-e^{0.04}}{e^{-0.06}-e^{0.04}}q_+.$$

We have that Q is an EMM as $\mathbb{Q}[A] > 0 \iff \mathbb{P}[A] > 0$ for all $A \in \mathcal{F}_t^S$.

We can for example pick a choice of $(q_+, q_-) = (0.2, 0.2210)$ for Q

We start by constructing the trinomial tree for the stock price, from $S_0 = 10$

				17.507
			15.220	15.841
		13.231	13.771	14.333
	11.503	11.972	12.461	12.969
10.000	10.408	10.833	11.275	11.735
	9.418	9.802	10.202	10.618
		8.869	9.231	9.608
			8.353	8.694
				7.866

To build the trinomial tree for the put option, we need to calculate the payoff at each leaf node, given by $(25 - S_T)^+$

At $T = 4$, all payoffs are positive, being: $25 - 17.507 = 7.493, \cdots, 25 - 7.866 = 17.134$.

Each node at $T = 3$ is given by the expectation over Q of each of its possible payoffs $(K - S_4)^+$, discounted back to $T = 3$ with a factor of $\frac{1}{R} = e^{-0.04}$.

With our choice of $(q_+, q_-) = (0.2, 0.2210)$, the price of the put at the upper node at $t = 3$ is given by

$$e^{-0.04}\left(7.493q_+ + 9.159(1-(q_+ + q_-)) + 10.667q_-\right)$$
$$= e^{-0.04}\left(7.493 * 0.2 + 9.159 * 0.579 + 10.667 * 0.2210\right) = 8.800$$

and so on. We arrived with the following arbirage-free tree, which has the non-arbitrage price *of the put at each time under each possible stock price, under the risk-neutral measure Q:*

				7.493
			8.800	9.159
		9.847	10.248	10.667
	10.670	11.106	11.559	12.031
11.304	11.765	12.245	12.745	13.265
	12.755	13.276	13.818	14.382
		14.209	14.789	15.392
			15.667	16.306
				17.134

We additionally note that the whole tree is independent of our choice of (q_+, q_-)*, i.e. if we use* $(0.1, 0.111)$*, we will get the same tree.*

The reasons behind is this:

We have showed that any (q_+, q_-) *that satisfies* $q_+ > 0, q_- > 0, 1 - q_+ - q_- > 0$ *and*

$$\frac{1}{R}\left(Uq^+ + M(1 - (q^+ + q^-)) + Dq^-\right) = 1$$

is an EMM.

Given the possible payoffs $P(U), P(M), P(D)$ *at* $t+1$*, where* $P(x) = (K - S_t x)^+$*, we have that the price of the put option at* t *is given by*

$$\Pi_t(S_t) = \frac{1}{R}\left(q^+ P(U) + (1 - (q^+ + q^-))P(M) + q^- P(D)\right)$$

In general this should differs with different choices of (q_+, q_-)*,*

But we note that our interested payoff is a deep ITM put option, with $K > S_0 U^4 > S_0 M^4 > S_0 D^4$*, so we have that at* $t = T - 1$*:* $P(U) = (K - S_t U)^+ = K - S_t U$*, and vice versa for* $P(M), P(D)$*,*

$$\begin{aligned}
\Pi_t(S_t) =& \frac{1}{R}\left(q^+(K - S_t U) + (1 - (q^+ + q^-))(K - S_t M) + q^-(K - S_t D)\right) \\
=& \frac{1}{R}\left(K \cdot 1 - S_t(Uq^+ + M(1 - (q^+ + q^-)) + Dq^-)\right) \\
=& \frac{K}{R} - S_t \frac{1}{R}\left(Uq^+ + M(1 - (q^+ + q^-)) + Dq^-\right) \\
=& \frac{K}{R} - S_t
\end{aligned}$$

so the price of the put option at $T-1$ is independent of our choice of (q_+, q_-).

Now we consider $t = T-2$:

the potential payoffs at $T-1$ is $\frac{K}{R} - S_tU$, $\frac{K}{R} - S_tM$, $\frac{K}{R} - S_tD$,

So at $t = T-2$:

$$\begin{aligned}\Pi_t(S_{T-2}) =& \frac{1}{R}\left(q^+\left(\frac{K}{R} - S_tU\right) + (1-(q^+ + q^-))\left(\frac{K}{R} - S_tM\right) + q^-\left(\frac{K}{R} - S_tD\right)\right)\\ =& \frac{1}{R}\left(\frac{K}{R} - S_t(Uq^+ + M(1-(q^+ + q^-)) + Dq^-)\right)\\ =& \frac{K}{R^2} - S_t\end{aligned}$$

Recursively, we note that our put has the same price as a short position in forward contract at each t,

i.e.

$$\Pi_t(S_t) = Ke^{-r(T-t)} - S_t$$

We can check that at $t = 0$, we can discount K to $Ke^{-rT} = 25e^{-0.04 \cdot 4} = 21.304$ to get $\Pi_0 = Ke^{-rT} - S_0 = 11.304$.

1.3 Change of Measure

We have examined changing measure from $\mathbb{P}$ to $\mathbb{Q}$, now we should explore more general changes of measure.

Theorem 1.6 (Radon-Nikodym Theorem). *If $q \ll \mu$, then there exists a unique measurable function $f : X \to [0, \infty]$:*

$$dq = f d\mu$$

or equivalently,

$$q(A) = \int_A f \, d\mu, \quad \forall A \in \mathcal{F}.$$

where f is the ***Radon-Nikodym derivative****: :*

$$f = \frac{dq}{d\mu}.$$

We can compute the expecation under Q using the Radon-Nikodym derivative $\frac{dQ}{dP}$:

$$\begin{aligned} E_Q[X] &= \int X \, dQ \\ &= \int X \frac{dQ}{dP} \, dP \\ &= E_P\left[\frac{dQ}{dP} X\right] \end{aligned}$$

Definition 1.6. A numeraire is a nonnegative random variable S_t such that $S_t > 0$ a.s.
In general, this can be any non-dividend paying tradable asset.
Given a numeraire S_t, a measure $\mathbb{Q}^{\mathbb{S}}$ is such that Y/S is a martingale under $\mathbb{Q}^{\mathbb{S}}$, where Y is any other tradable asset. (also non-dividend paying)

We note that when the numeraire is B_t, we have $\mathbb{Q} = \mathbb{Q}^{\mathbb{B}}$.
Say under $\mathbb{Q}^{\mathbb{B}}$, some tradable asset S has this SDE:

$$dS_t = \mu^B dt + \sigma^B dW_t^B$$

Where μ^B, σ^B are functions of t, S_t.
We have by Leibniz's rule:

$$d\frac{S_t}{B_t} = S_t d(\frac{1}{B_t}) + \frac{1}{B_t} dS_t + dS_t d(\frac{1}{B_t})$$

Notice the quadratic variation term $dS_t d(1/B_t)$ is 0, as this is a multiple of $d_t dW_t$, which is 0. So we have:

$$\begin{aligned} d\frac{S_t}{B_t} =& S_t d(e^{-rt}) + \frac{1}{B_t} dS_t \\ =& S \cdot (-r) e^{-rt} dt + e^{-rt} \left(\mu^B dt + \sigma^B dW_t\right) \\ =& e^{-rt} \left(S \cdot (-r) dt + \mu dt + \sigma dW\right) \\ =& \frac{1}{B_t} \left(\left(\mu - r S_t\right) dt + \sigma dW_t\right) \end{aligned}$$

Since S_t/B_t is a martingale, which should have zero drift, we must have $\mu = rS_t$.
i.e. Under $\mathbb{Q}^{\mathbb{B}}$, all tradable assets have drifts equal to r times the asset price, which is precisely the definition of the risk-neutral measure.

Theorem 1.7 (Girsanov's Theorem)**.** *Let W_t be a Brownian motion under the measure $\mathbb{P}$, and let θ_t be a predictable process.*

Define the process:

$$Z_t = \exp\left(\int_0^t \theta_s dW_s - \frac{1}{2}\int_0^t \theta_s^2 ds\right)$$

Then, under the measure $\mathbb{Q}$ *defined by the Radon-Nikodym derivative:*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_T$$

the process $W_t - \int_0^t \theta_s ds$ *is a Brownian motion under the measure* $\mathbb{Q}$*.*

1.4 Stochastic Integral

Consider the following SDE:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

We note that $t \mapsto W_t$ is continuous but nowhere differentiable.
In other words, dW_t cannot be interpreted as a time differential.

Definition 1.7. Let f and g be functions defined on the interval $[a, b]$, where f is a continuous function and g is a function of bounded variation on $[a, b]$. The **Riemann-Stieltjes Integral** of f with respect to g over $[a, b]$ is defined as:

$$\int_a^b f(x)\, dg(x) = \lim_{||P|| \to 0} \sum_{i=1}^n f(\xi_i)\Delta g_i$$

whenever this limit exists. Here,

- $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a, b]$, - $\xi_i \in [x_{i-1}, x_i]$ is a sample point, - $\Delta g_i = g(x_i) - g(x_{i-1})$, and - $||P||$ is the norm of the partition, given by the length of the largest subinterval.

We note that by definition, W_t has unbounded variation.
Given a path $t \mapsto W_t(\omega)$, the integral:

$$\int_0^t f(s, X_s(\omega))dW_s(\omega)$$

does not exist as a **Riemann-Stieltjes Integral**.

Proof. If it can be defined as a **Riemann-Stieltjes Integral**, this limit should converge:

$$\int_0^T \sigma(X_s)dW_s = \lim_{||P||\to 0} \sum_{i=1}^{n} \sigma(X_{\xi_i})(W_{t_{i+1}} - W_{t_i})$$

for any partition $P = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ and any sample points $\xi_i \in [t_i, t_{i+1}]$.

This does not work for brownian motion, as the variation of W_t is unbounded.

More precisely, for a partition P_n, define the upper and lower bound:

$$U(f, P_n) = \sum_{i=1}^{n} \sup_{t\in[t_{i-1},t_i]} f(t, X_t)(W_{t_{i+1}} - W_{t_i})$$
$$L(f, P_n) = \sum_{i=1}^{n} \inf_{t\in[t_{i-1},t_i]} f(t, X_t)(W_{t_{i+1}} - W_{t_i})$$

the Riemann Integral exists iff:

$$U(f, P_n) - L(f, P_n) \to 0$$

□

Definition 1.8. The **Ito Integral** uses the left point of each subinterval to define the integral:

$$\int_0^T f(s, X_s)dW_s = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i, X_{t_i})(W_{t_{i+1}} - W_{t_i})$$

Definition 1.9. The **Stratonovich Integral** uses the midpoint of each subinterval to define the integral:

$$\int_0^T f(s, X_s) \circ dW_s = \lim_{n\to\infty} \sum_{i=1}^{n} f(\frac{t_i + t_{i+1}}{2}, X_{\frac{t_i+t_{i+1}}{2}})(W_{t_{i+1}} - W_{t_i})$$

Remark. Stratonovich looks into the future, while Ito does not.

This is because as we are evaluating at t_i, Ito evaluates $f(X)$ at t_i, while Stratonovich evaluates $f(X)$ at $\frac{t_i+t_{i+1}}{2}$.

Note clearly if $f(t, X_t)$ is a deterministic function of t, then the two integrals are the same.

We know that

$$E[W] = 0$$
$$E[W_t^2] = t$$
$$W_T - W_t \sim N(0, T - t)$$

More formally, in the Ito integral, we have:

$$E\left[\int_0^T f(s, X_s)dW_s\right] = 0$$

$$E\left[(\int_0^T f(s, X_s)dW_s)^2\right] = \int_0^T E\left[f^2(s, X_s)\right] ds$$

1.5 Existence and Uniqueness of Solutions

A sufficient condition for the existence and uniqueness of solutions to the SDE:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

is that μ and σ are measurable, Lipschitz continuous in x and locally bounded in t.

This is known as the **Kunita-Watanabe Theorem**.

More formally, we need Global Lipschitz Continuity for some K:

$$|\mu(t, x) - \mu(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq K|x - y| \ \forall x, y \in \mathbb{R}, \ \forall t \in [0, T]$$

and Local Boundedness / Linear Growth for some K:

$$|\mu(t, x)| + |\sigma(t, x)| \leq K(1 + |x|) \ \forall x \in \mathbb{R}, \ \forall t \in [0, T]$$

1.6 Itô's Formula

Theorem 1.8 (Itô's Formula)**.** *Let X_t be a stochastic process satisfying the SDE:*

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

and let $f(t, x)$ be a function of class $C^{2,1}$, i.e. f is twice continuously differentiable in x and once continuously differentiable in t.

Then, the process $Y_t = f(t, X_t)$ satisfies the SDE:

$$dY_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dX_t^2$$

Or equivalently:

$$dY_t = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma\frac{\partial f}{\partial x}dW_t$$

Where we note that

$$dt^2 = 0$$
$$dt \cdot dW_t = 0$$
$$dW_t^2 = dt$$

2 Interest Rate Products

2.1 Zero Coupon Bonds

Definition 2.1. A **stochastic discount factor** $D(t,T)$ is the price at time t of receiving 1 at time T.

Consider a deterministic bond B_t with

$$dB_t = r_t B_t dt$$

then clearly

$$B_t = B_0 \cdot e^{\int_0^t r_s ds}$$

letting $B_T = 1$, we have $B_0 = e^{-\int_0^T r_s ds} = D(0,T)$, so the stochastic discount factor is the price of a riskless bond that pays 1 at T.

Definition 2.2. A **Zero Coupon Bond** is a bond that pays its Face Value at T. WLOG assume the face value is 1, then its price at t is given by

$$P(t,T) = E_Q\left[e^{-\int_t^T r_t dt}\right] = E_Q\left[D(t,T)\right] \tag{2.1}$$

where r_t is the short rate at time t, and the expectation is taken with respect to the risk free measure Q.

Note in the case when r_t is constant, we have

$$P(t,T) = E_Q\left[e^{-r(T-t)}\right]$$
$$= e^{-r(T-t)} = D(t,T)$$

2.2 Linear, Compound, Short rates and Annual Spot Rates

Definition 2.3. The **Spot Linear / Simply Compounded / LIBOR Rate** L is the rate that pays linear interest.

Basically when you pay X, you are supposed to receive $X(1 + L \cdot n)$ after n years

To receive 1 at T, while paying $P(t,T)$ at time t, so the spot rate $L(t,T)$ is given by

$$\begin{aligned} p(t,T)(1 + L \cdot (T-t)) &= 1 \\ L(t,T) &= \frac{\frac{1}{P(t,T)} - 1}{T-t} \\ &= \frac{1 - P(t,T)}{P(t,T) \cdot (T-t)} \end{aligned}$$

Definition 2.4. The **Spot Compound / Continuous Rate** $R(t,T)$ is the rate that pays continuous interest

Pay X at t and receive $X \cdot e^{R \cdot (T-t)}$ at T

To receive 1 at T, while paying $P(t,T)$ at time t:

$$P(t,T) \cdot e^{R \cdot (T-t)} = 1$$

so the spot rate $R(t,T)$ is given by

$$R(t,T) = \frac{\log(1/P)}{T-t} = \frac{-\log P(t,T)}{T-t}$$

Note here we have the short rate $r_t = \lim_{t \to T^-} R(t,T) \approx R(t,t)$

Definition 2.5. With $R(t,T)$, we arrive with the **short rate** r_t:

$$\begin{aligned} r_t &= \lim_{T \to t} R(t,T) \\ &= \lim_{T \to t} \frac{-\log P(t,T)}{T-t} \end{aligned}$$

Definition 2.6. The **Annual Spot Rate** Y is the rate that pays interest annually.

Pay X at t and receive $X \cdot (1+Y)^{T-t}$ at T

We again solve

$$P \cdot (1+Y)^{T-t} = 1$$

So the spot rate $Y(t,T)$ is given by

$$Y(t,T) = \frac{1}{P^{\frac{1}{T-t}}} - 1$$

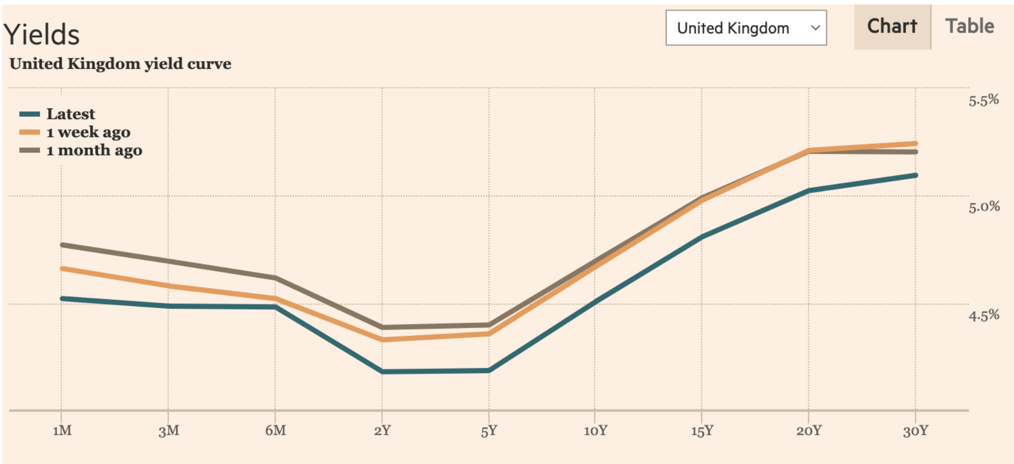

Figure 1: Yield Curve

In this case the zero coupon bond price is given by

$$P(t,T) = \frac{1}{(1+Y)^{T-t}}$$

Definition 2.7. The **Zero-Coupon Curve** is the curve given by

$$T \mapsto \begin{cases} L(t,T), & T \leq t+1, \\ Y(t,T), & T > t+1 \end{cases}$$

Basically, we use the LIBOR rate for expiry within one year, and the annual spot rate for long term rates.

An example of a yield curve in Feb 2025 is given by Figure 1, where we note that the yield initially goes down, but goes up when maturity $> 5Y$.

Note the general bond formula for a maturity of N years is given by

$$P(0,N) = \sum_{n=1}^{N} \frac{C}{(1+Y)^n} + \frac{F}{(1+Y)^N}$$

Where C is the coupon payment, and F is the face value.

For fixed C and F, when price goes down, yield goes up.

The fact that less people demand long term bonds, causes the yield to go up, as investors expect rates to go up in the long run, hence demand higher yield for long term bonds.

2.3 Forward Rate

A **Forward Rate Agreement** (FRA) is a contract between two parties, where one party pays a fixed interest rate and the other party pays a floating interest rate. The contract is settled at the end of the contract period.

As we consider at t, the FRA(T_1, T_2) is a contract that starts at T_1 and ends at T_2. The fixed rate is agreed at T_1, and settled at T_2.
For the period $\big(T_1, T_2\big)$, the holder of the **Payer FRA** pays fixed $K \cdot \big(T_2 - T_1\big)$ and receives floating $L(T_1, T_2) \cdot \big(T_2 - T_1\big)$.
The holder of the **Receiver FRA** has the opposite payoff, with payoff at T_2 given by

$$\big(K - L(T_1, T_2)\big)\,\big(T_2 - T_1\big)\,N,$$

where N is the notional amount, and we denote $\tau = T_2 - T_1$.
Holding 1 at T_2 is equivalent to holding $P(T_1, T_2)$ at T_1, and holding $P(t, T_1)P(T_1, T_2)$ at t.
So we can use $P(t, T_1)\,P(T_1, T_2)$ as the discount factor from T_2 to t.
Note that $P(T_1, T_2)$ is random at t, and is denoted as the **Forward Price** $P(t, T_1, T_2)$
Recall that we have

$$P(T_1, T_2)\left(1 + L(T_1, T_2)\tau\right) = 1$$
$$L(T_1, T_2) = \frac{1}{\tau}\left(\frac{1}{P(T_1, T_2)} - 1\right)$$

Where $P(T_1, T_2)$ is random at t.
Note that we have $D(T_1, T_2) = D(t, T_2)/D(t, T_1)$, but the same doesn't hold for the zero coupon bond price P.
Nevertheless to price the FRA, we estimate $P(T_1, T_2)$ with $P(t, T_1)P(T_1, T_2)$, and have

$$\hat{L}(T_1, T_2) = \frac{1}{\tau}\left(\frac{P(t, T_1)}{P(t, T_2)} - 1\right)$$

Definition 2.8. To make the FRA fair, we need

$$K \;=\; \hat{L}(T_1, T_2) \;=\; \frac{1}{\tau}\left(\frac{P(t, T_1)}{P(t, T_2)} - 1\right). \tag{2.2}$$

Which is computable (non-random) at t.
This is the **Forward Linear Rate** $F(t, T_1, T_2)$.

Remark.

$$P(t, T_1)P(T_1, T_2) = E[D(t, T_1)]E[D(T_1, T_2)] \neq E[D(t, T_1)D(T_1, T_2)] = E[D(t, T_2)] = P(t, T_2)$$

As we have no evidence that $D(t, T_1)$ and $D(T_1, T_2)$ are uncorrelated.

Definition 2.9. Similarly, we can define the **Forward Compound Rate** $R(t, T_1, T_2)$:
By letting $K = R(T_1, T_2)$ instead of $K = L(T_1, T_2)$
Recall

$$R(T_1, T_2) = \frac{-\log P(T_1, T_2)}{T_2 - T_1}$$

Writing $-\log P(T_1, T_2) = -\log P(t, T_1) + \log P(t, T_2)$, we have

$$R(t, T_1, T_2) \;=\; \frac{-\log P(t, T_1) + \log P(t, T_2)}{T_2 - T_1}. \tag{2.3}$$

But we continue with the linear / LIBOR rate for now.

2.4 FRA

Note we derived the forward linear and compound rate above informally, by plugging in $P(t, T_1)P(T_1, T_2)$ for $P(t, T_2)$ in the definition of $L(T_1, T_2)$.
This is not rigorous, so we formally derive the price of FRA below, and derive the forward rate by setting FRA to 0.

Definition 2.10. Assume N = 1. The payoff of the **Receiver FRA**(t, T_1, T_2) at T_2 is $\tau\big(K - L(T_1, T_2)\big)$

To price it at t, we discount the payoff with the factor $D(t, T_2)$, and take the expectation under Q

$$\begin{aligned}\text{Receiver FRA}(t) &= \tau\, E_Q\big[\, D(t, T_2)\,\big(K - L(T_1, T_2)\big)\big] \\ &= \tau\, K\, E_Q\big[\, D(t, T_2)\big] - \tau\, E_Q\big[\, D(t, T_2)\, L(T_1, T_2)\big] \\ &= \tau\, K\, P\big(t, T_2\big) - \tau\, E_Q\big[\, D(t, T_2)\, L(T_1, T_2)\big]\end{aligned}$$

We note that the second term:

$$\begin{aligned}\tau\, E_Q\big[\, D(t, T_2)\, L(T_1, T_2)\big] &= \tau\, E_Q\big[\, D(t, T_1) D(T_1, T_2)\, L(T_1, T_2)\big] \\ &= \tau\, E_Q\big[E_T[D(t, T_1) D(T_1, T_2)\, L(T_1, T_2)]\big]\end{aligned}$$

Where we denote E_T as the expectation taken at T_1, (i.e. with the filtration $\mathcal{F}_{T_1}$).
We note that at T_1, $D(t, T_1)$ and $L(T_1, T_2)$ are non-random. Therefore

$$\begin{aligned}\tau\, E_Q\big[\, D(t, T_2)\, L(T_1, T_2)\big] &= \tau\, E_Q\big[D(t, T_1) L(T_1, T_2) E_T[D(T_1, T_2)]\big] \\ &= \tau\, E_Q\big[D(t, T_1) L(T_1, T_2) P(T_1, T_2)\big]\end{aligned}$$

$$
\begin{aligned}
&= \tau\, E_Q\big[D(t,T_1)\frac{1}{\tau}\left(\frac{1}{P(T_1,T_2)} - 1\right) P(T_1,T_2)\big] \\
&= E_Q\big[D(t,T_1)\,(1 - P(T_1,T_2))\big] \\
&= P(t,T_1) - E_Q\big[D(t,T_1)P(T_1,T_2)\big]
\end{aligned}
$$

Again, note that the second term

$$
\begin{aligned}
E_Q\big[D(t,T_1)P(T_1,T_2)\big] &= E_Q\big[D(t,T_1)E_{T_1}[D(T_1,T_2)]\big] \\
&= E_Q\big[E_{T_1}[D(t,T_1)D(T_1,T_2)]\big] \\
&= E_Q\big[D(t,T_2)\big] \\
&= P(t,T_2)
\end{aligned}
$$

So we end up with

$$
\text{Receiver FRA}(t) = \tau\, K\, P\big(t,T_2\big) - P(t,T_1) + P(t,T_2) \tag{2.4}
$$

We solve FRA(K) = 0, and note that the solution is exactly $K = F(t,T_1,T_2) = \frac{1}{\tau}\left(\frac{P(t,T_1)}{P(t,T_2)} - 1\right)$:

$$
FRA(K) = \Big(\tau\, K\, P(t,T_2) - P(t,T_1) + P(t,T_2)\Big) = 0
$$

$$
K = \frac{1}{\tau}\left(\frac{P(t,T_1) - P(t,T_2)}{P(t,T_2)}\right)
$$

Definition 2.11. The **Instantaneous Forward Rate** $f(t,T)$ is the estimated rate to be paid from T to T + dt, computed by taking the following limit

$$
\begin{aligned}
f(t,T) &= \lim_{\tau\to 0} F(t,T,T+\tau) = \lim_{\tau\to 0} \frac{P(t,T) - P(t,T+\tau)}{P(t,T+\tau)\cdot\tau} \\
&= \lim_{\tau\to 0} -\frac{1}{P(t,T+\tau)} \cdot \frac{P(t,T+\tau) - P(t,T)}{\tau}
\end{aligned}
$$

We note that both terms in the limit has well defined limits, so the limit of the product is the product of the limits. So

$$
\begin{aligned}
f(t,T) &= -\frac{1}{P(t,T)} \cdot \frac{\partial}{\partial T} P(t,T) \\
&= -\frac{\partial}{\partial T} \log P(t,T)
\end{aligned}
$$

This can be viewed as an estimate for the future short rate $r(T)$, which is random at t.

Note under mild regularity conditions, the instantaneous forward rate should converge to the short rate:

$$
\begin{aligned}
r_t &= \lim_{\tau \to 0} f(t, t+\tau) \\
&= \lim_{\tau \to 0} -\frac{\partial}{\partial T} \log P(t, T)
\end{aligned}
$$

While we previously have

$$
\begin{aligned}
r_t &= \lim_{\tau \to 0} R(t, t+\tau) \\
&= \lim_{\tau \to 0} -\frac{1}{T} \log P(t, T)
\end{aligned}
$$

Consider a trivial example of a quadratic zero coupon bond.

$$
P(0, T) = e^{-AT^2 - BT - C}
$$

First we note that we need $P(0,0) = 1$, so $C = 0$.

Additionally we need $P(0,T) \in [0,1)$, so $A \geq 0\,, B \geq 0$.

Then we have

$$
\begin{aligned}
R(0, T) &= -\frac{1}{T}(-AT^2 - BT) = AT + B \\
r_0 &= R(0,0) = B
\end{aligned}
$$

And

$$
\begin{aligned}
f(0, T) &= -\frac{\partial}{\partial T} \log P(0, T) = 2AT + B \\
r_0 &= f(0,0) = B
\end{aligned}
$$

So in this case, the instantaneous forward rate and continuous Compounded rate agrees in the limit to the short rate

2.5 Remarks: Forward Rate is an biased estimator

Note in (2.1) and (2.2), we showed

$$
\tau\, E_Q\Big[D\big(t, T_2\big)\,\big(K - L(T_1, T_2)\big)\Big] \;=\; \tau\, K\, P\big(t, T_2\big) \;-\; P\big(t, T_1\big) \;+\; P\big(t, T_2\big).
$$

Where

$$E_Q\Big[D\big(t,T_2\big)\,\big(K - L(T_1,T_2)\big)\Big] = K\,P\big(t,T_2\big) - E_Q\Big[D\big(t,T_2\big)\,L(T_1,T_2)\Big].$$

Hence

$$\begin{aligned} P\big(t,T_1\big) - P\big(t,T_2\big) =& \tau\Big(\,K\,P\big(t,T_2\big)\ -\ E_Q\big[D\big(t,T_2\big)\,(K-L)\big]\Big) \\ =& \tau\Big(K\,P\big(t,T_2\big) - K\,P\big(t,T_2\big) + E_Q\big[D\big(t,T_2\big)\,L(T_1,T_2)\big]\Big) \\ =& \tau\ E_Q\Big[D\big(t,T_2\big)\,L(T_1,T_2)\Big] \end{aligned}$$

So we have

$$\frac{P\big(t,T_1\big) - P\big(t,T_2\big)}{\tau}\ =\ E_Q\Big[D\big(t,T_2\big)\,L(T_1,T_2)\Big].$$

Note the LHS is also equal to $F(t,T_1,T_2)\,P(t,T_2)$. Thus

$$F(t,T_1,T_2)\ P(t,T_2)\ =\ E_Q\Big[D\big(t,T_2\big)\,L(T_1,T_2)\Big].$$

Where we had $P(t,T_2) = E_Q\big[D(t,T_2)\big]$, so in general $F(t,T_1,T_2) \neq E_Q\big[L(T_1,T_2)\big]$. They would only be equal if $D(t,T_2)$ and $L(T_1,T_2)$ were uncorrelated, which is not generally true.

Similarly we have that the instantaneous forward rate is an biased estimator for the short rate:

$$f(t,T) \neq E_Q\big[r(T)\big]$$

2.6 Interest Rate Swaps

Definition 2.12. An **Interest Rate Swap** is a contract between two parties, where FRA are exchanged at future dates $T_\alpha, T_{\alpha+1}, \ldots, T_\beta$.
We define the time intervals $\tau = (\tau_{\alpha+1}, \ldots, \tau_\beta)$, where $\tau_i = T_i - T_{i-1}$.
The **Receiver Swap (RFS)** has payoff at each payment date given by

$$N\tau_i(K - L(T_{i-1},T_i))$$

and the **Payer Swap (PFS)** has the opposite payoff.
This is simply a sum of FRAs at each (T_{i-1}, T_i):

$$\text{RFS(t)} = \sum_{i=\alpha+1}^{\beta} FRA(t, T_{i-1}, T_i, \tau_i)$$

$$= \sum_{i=\alpha+1}^{\beta} \tau_i \cdot P(t,T_i)\,(K - F(t,T_{i-1},T_i))$$
$$= \sum_{i=\alpha+1}^{\beta} \tau_i \cdot P(t,T_i)\left(K - \frac{P(t,T_{i-1}) - P(t,T_i)}{P(t,T_i)\cdot \tau_i}\right)$$

Note that this is equivalent to plugging in the sum of FRA prices: (2.4)

$$\begin{aligned}\text{RFS(t)} &= \sum_{i=\alpha+1}^{\beta} \Big(\tau_i P(t,T_i)K - P(t,T_{i-1}) + P(t,T_i)\Big) \\ &= \Big(\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)K\Big) - P(t,T_\alpha) + P(t,T_\beta)\end{aligned} \tag{2.5}$$

Again we plug in RFS(K) = 0 to get the fair rate K, or the **Forward Swap Rate**:

$$\sum_{i=\alpha+1}^{\beta} \tau_i P(t,T_i)K = P(t,T_\alpha) - P(t,T_\beta)$$
$$\Rightarrow K = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum \tau_i \cdot P(t,T_i)}$$

Definition 2.13. The **Forward Swap Rate** $S_{\alpha,\beta}(t)$ is the rate that makes the interest rate swap a fair contract:

$$S(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum \tau_i \cdot P(t,T_i)}$$

The Receiver Swap (2.5) can be rewritten as:

$$\begin{aligned}\text{RFS(t)} =& N\Big(\sum \tau_i P(t,T_i)K - \big(P(t,T_\alpha) - P(t,T_\beta)\big)\Big) \\ =& N\sum \tau_i P(t,T_i)\Big(K - S(t)\Big)\end{aligned}$$

Remark. Note that $S_{\alpha,\beta}(t)$ is simply a weighted average of the forward rates:

$$S(t) = \sum w_i F_i$$

Where

$$\begin{aligned}w_i =& \frac{\tau_i P(t,T_i)}{\sum \tau_i P(t,T_i)} \\ F_i =& F(t,T_{i-1},T_i) = \frac{P(t,T_{i-1}) - P(t,T_i)}{\tau_i P(t,T_i)}\end{aligned}$$

2.7 Caps/Floors

Definition 2.14. A **Caplet** is a European call options on interest rate from T_{i-1} to T_i, with payoff at T_i given by

$$N \cdot \tau_i \left(L(T_{i-1}, T_i) - K\right)^+$$

And a **Cap** is simply a sum of Caplets:

$$N \sum_{i=\alpha+1}^{\beta} \tau_i \left(L(T_{i-1}, T_i) - K\right)^+$$

We discount each payment back to t with $D(t, T_i)$, and take the expectation under Q to price it at t:

$$\begin{aligned} \text{Cap(t)} =& E_Q \left[N \sum_{i=\alpha+1}^{\beta} \tau_i \cdot D(t, T_i) \cdot \left(L(T_{i-1}, T_i) - K\right)^+ \right] && (2.6) \\ =& N \sum_{i=\alpha+1}^{\beta} \tau_i \cdot P(t, T_i) \cdot Caplet(T_i, \tau_i, K) && (2.7) \end{aligned}$$

To price the Caplet, we need to use Black's formula

$$\begin{aligned} Caplet = Black(K, F, \sigma) &= F\phi(d_1) - K\phi(d_2) \\ d_1 &= \frac{\ln(F/K) + \frac{\sigma^2 T_{i-1}}{2}}{\sigma\sqrt{T_{i-1}}} \\ d_2 &= \frac{\ln(F/K) - \frac{\sigma^2 T_{i-1}}{2}}{\sigma\sqrt{T_{i-1}}} \end{aligned}$$

$$\text{Cap(t)} = N \sum_{i=\alpha+1}^{\beta} \tau_i \cdot P(t, T_i) \cdot Black(K, F(t, T_{i-1}, T_i), \sigma)$$

Here σ is the implied volatility retrived from market quotes in $[T_\alpha, T_\beta]$, and $F(t, T_{i-1}, T_i)$ is the forward rate.

Similarly for **Floorlet**, or the put option:

$$Floorlet = Black(K, F, \sigma) = -F\phi(-d_1) + K\phi(-d_2)$$

Recall the standard Black-Scholes:

$$\begin{aligned}C =& S_0N(d_1) - Ke^{-rT}N(d_2),\\ P =& Ke^{-rT}N(-d_2) - S_0N(-d_1),\\ d_1 =& \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},\\ d_2 =& d_1 - \sigma\sqrt{T}.\end{aligned}$$

Remark. A cap is ATM iff its strike is the **Forward Swap Rate** $S(t)$.

It is ITM if $K < S(t)$, and OTM if $K > S(t)$.

Remark. Similar to Call-Put parity, we have

$$\begin{aligned}\text{Caplet}(K) - \text{Floorlet}(K) =& F\phi(d_1) - K\phi(d_2) - (-F\phi(-d_1) + K\phi(-d_2))\\ =& F - K\end{aligned}$$

Which is the payoff of a single payer FRA

We notice for ATM Caplet and Floorlet with $K = F(t, T_{i-1}, T_i)$, we have $Caplet = Floorlet$

For ATM Cap and Floor with $K = S(t)$, we have $Cap = Floor$

This is similar to the Call-Put parity, where we have

$$C - P = S_0 - Ke^{-rT}$$

And when $K = S_0e^{rT}$, we have $C = P$.

2.8 Swaptions

To price swaptions, note that the market Black's formula for caplets/floorlets, consistent with the LMM, is not consistent with the market Black's formula for swaptions, which is consistent with the SMM.

To price swaptions under LMM, one has to use Monte Carlo simulations

2.8.1 Swap Market Model

SWAPTIONS can be managed well in the LIBOR model only through approximations like drift freezing. To properly deal with swaptions, one would have to use the SMM

Recall payoff of swaption at T_α is

$$(S_{\alpha,\beta}(T_\alpha) - K)^+ \sum_i \tau_i P(T_\alpha, T_i)$$

To price it at $t = 0$, we add a discount factor e^{-T_α}, and take risk-neutral expectation:

$$E_Q \left[(S_{\alpha,\beta}(T_\alpha) - K)^+ \sum_i \tau_i P(T_\alpha, T_i) \frac{B_0}{B_{T_\alpha}} \right]$$

We choose the numeraire $C_{\alpha,\beta}(t)$ as:

$$C_{\alpha,\beta}(t) = \sum_i \tau_i P(t, T_i)$$

with the related measure be $Q^{\alpha,\beta}$, under which any price process divided by $C_{\alpha,\beta}(t)$ is a martingale

We note that in this case, the forward swap rate $S_{\alpha,\beta}(t)$ is a martingale:

$$\begin{aligned} S(t) =& \frac{P(t, T_\alpha) - P(t, T_\beta)}{\sum \tau_i \cdot P(t, T_i)} \\ =& \frac{P(t, T_\alpha) - P(t, T_\beta)}{C_{\alpha,\beta}(t)} \end{aligned}$$

SO that it has zero drift GBM under $Q^{\alpha,\beta}$:

$$dS(t) = \sigma S(t) dW_t^{\alpha,\beta}$$

Applying change of numeraire:

$$\begin{aligned} E_Q \left[(S_{\alpha,\beta}(T_\alpha) - K)^+ C_{\alpha,\beta}(T_\alpha) \frac{B_0}{B_{T_\alpha}} \right] =& E_{Q_{\alpha,\beta}} \left[(S_{\alpha,\beta}(T_\alpha) - K)^+ C_{\alpha,\beta}(T_\alpha) \frac{C_0}{C_{\alpha,\beta}(T_\alpha)} \right] \\ =& C_{\alpha,\beta}(0) E_{Q_{\alpha,\beta}} \left[(S_{\alpha,\beta}(T_\alpha) - K)^+ \right] \end{aligned}$$

Under this setting, we can simply apply Black's formula:

$$\begin{aligned} Swaption(0) = Black(S, F, \sigma) &= C_{\alpha,\beta}(0) \Big[S_{\alpha,\beta}(0) \phi(d_1) - K \phi(d_2) \Big] \\ &= \sum_i \tau_i P(0, T_i) \Big(S_{\alpha,\beta}(0) \phi(d_1) - K \phi(d_2) \Big) \end{aligned}$$

Remark. SMM is the only model that is consistent with this market formula.

LMM is not compatible with the Black formula for Swaptions.

2.9 Swaption Under LMM

3 One Factor Short Rate Models

3.1 Vasicek Model

$$
\begin{aligned}
dr_t &= k(\theta - r_t)dt + \sigma dW_t \\
r_t &= r_0 e^{-kt} + \theta(1 - e^{-kt}) + \sigma e^{-kt} \int_0^t e^{ks} dW_s
\end{aligned}
$$

We know that k is speed of mean reversion, θ is the long term mean, and σ is the volatility of the short rate.

The drift term $k(\theta - r_t)dt$ is linear in r_t, and the diffusion term σdW_t is linear in dW_t, suggesting the rates is normally distributed, meaning it can go negative.
To compute the variance:

$$
\begin{aligned}
Var(r_t) &= Var\left(\sigma e^{-kt} \int_0^t e^{ks} dW_s\right) \\
&= E\left[\left(\sigma \int_0^t e^{k(t-s)} dW_s\right)^2\right] - E\left[\sigma \int_0^t e^{k(t-s)} dW_s\right]^2 \\
&= \sigma^2 \int_0^t e^{2k(t-s)} ds - 0 \\
&= \frac{\sigma^2}{2k}(1 - e^{-2kt})
\end{aligned}
$$

So we have $r_t \sim N(A, B)$, where

$$
\begin{aligned}
A =& r_0 e^{-kt} + \theta(1 - e^{-kt}), \\
B =& \frac{\sigma^2}{2k}(1 - e^{-2kt})
\end{aligned}
$$

We can see that as $t \to \infty$, $A \to \theta$, and $B \to \frac{\sigma^2}{2k}$, i.e. for larger k, the convergence is faster, and smaller volatiltiy in the long run.

More generally, this is a special case of the Ornstein-Uhlenbeck process,

$$dx_t = \alpha(\mu - x_t)dt + \sigma_t dW_t$$

where x_t is the state variable, α is the speed of mean reversion.

Some of the problems with this model are:

- The short rate can go negative,
- Gaussian has thinner tails than market implied distributions,
- The model is endogenous, meaning it cannot reproduce the yield curve.

The CIR model is an improvement on the first two points.
To solve the third point, we can develop an exogenous model by making θ a function of time, $\theta(t)$, i.e.

$$dr_t = k(\theta_t - r_t)dt + \sigma dW_t$$

So we arrived with the one-factor Hull-White model.

3.2 CIR Model

For $r_0 > 0$,

$$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t$$

ensures positive interest rates with the Feller condition $2k\theta > \sigma^2$, (otherwise hits 0).
Again we compute the mean and variance:

$$E[r_t] = r_0 e^{-kt} + \theta(1 - e^{-kt})$$
$$Var(r_t) = \frac{\sigma^2}{2k}(1 - e^{-2kt})\theta + r_0 e^{-kt}\frac{\sigma^2}{k}(1 - e^{-kt})$$

In this case r_t has a non-central chi-squared distribution, has fatter tails than Gaussian, hence closer to market implied distributions.
It is less tractable, especially in in multi-factor extensions, where we need to add correlation between the factors.

A problem still, is that Gaussian distributions for the rates are not compatible with the market implied distributions as they have tails that are too thin.

In particular, both models are affine models.

$$P(t,T) = A(t,T)e^{-B(t,T)r_t}$$

Additionally, we work out the continuously compounded spot rate $R(t,T)$,
So

$$\begin{aligned} R(t,T) &= -\frac{\ln P(t,T)}{T-t} \\ &= -\frac{\ln A(t,T)}{T-t} - \frac{\ln e^{-B(t,T)r_t}}{T-t} \end{aligned}$$

So R is an affine transformation of r_t.
This suggests perfect terminal correlation:

$$\text{Corr}(R(t,T_1), R(t,T_2)) = 1$$

4 Libor Market Models

4.1 Forward Measures

We recall that the risk neutral measure Q is using the money market account $B_t = e^{rt}$ as numeraire.

Definition 4.1. T-Forward Measure

Let $T > t$ be a fixed time horizon. The forward measure $\mathbb{Q}^T$ uses the zero-coupon bond $P(t, T)$ as numeraire.
i.e. Let S_t be some price process, then $S_t/P(t, T)$ is a $\mathbb{Q}^T$-martingale.

$$\frac{S_t}{P(t,T)} = \mathbb{E}^{\mathbb{Q}^T}\left[\frac{S_T}{P(T,T)}\middle|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}^T}\left[S_T\middle|\mathcal{F}_t\right]$$

We can also use E_t^T to denote the expectation at t under $\mathbb{Q}^T$, then

$$V(t) = E_t^T[V(T)]P(t,T)$$

5 Post-LIBOR

5.1 Libor Scandal

In early 2010s, banks like Barclays, UBS, and RBS were found to have manipulated the LIBOR rate.
A team within a bank's Treasury or Money Market Desk should independently estimate the rate at which they can borrow from other banks, without influence from the trading desk.
The LIBOR rate is the average of the rates submitted by the banks, excluding the highest and lowest 25% of the submissions.
Traders were found to influence the submitters and help out each other, by submitting lower rates to make them look stronger (when they can't borrow at that rate), or higher rates when they hold payer IRS (and receive $L - K$).

By the 2010s, the volume of unsecured interbank lending had shrunk significantly. Banks were borrowing less from each other because:
Post-2008 regulations (Basel III) discouraged excessive interbank lending.
Banks relied more on secured funding (repo markets) rather than unsecured loans.
By the time regulators pushed for actual loan-based LIBOR submissions, there were only a few billion dollars of daily transactions, not reliable enough to serve as the benchmark for hundreds of trillions of dollars worth of financial products that it should.

In 2017, the FCA announced that LIBOR would be phased out by 2021, and replaced by the

SONIA rate in the UK, and the SOFR rate in the US.

- US: The Secured Overnight Financing Rate (SOFR)
- UK: The Sterling Overnight Index Average (SONIA)
- Eurozone: The Euro Short-Term Rate (ESTR) and EURIBOR continues to be used

Based on those rates, the new IRS is called overnight index swaps.

5.2 Collateralized vs. Uncollateralized Borrowing Rates

Definition 5.1. Collateralized Rate is the rate of loans backed by collateral.
The borrower pledges assets (e.g., government bonds) to secure the loan.

Examples include SOFR (US) Based on U.S. Treasury-backed repo transactions, and TONA (Tokyo Overnight Average Rate) based on Japanese government bonds.

On the other hand, both LIBOR and SONIA are uncollateralized rates.

Collateralized borrowing rates tend to be lower than uncollateralized rates because they have less credit risk.

5.3 Daily Compounded Rates

OIS are linked to daily rates, but exchange payments usually FOR 3M, 6M, 1Y, etc.

This is an important book in finance [1].
This is an important book in finance [2].

References

[1] Damiano Brigo and Fabio Mercurio. *Interest Rate Models - Theory and Practice*. 2nd. Springer Finance. Springer, 2006. ISBN: 978-3-540-22149-4.

[2] Arno van den Essen, Shigeru Kuroda, and Anthony J. Crachiola. *Polynomial Automorphisms and the Jacobian Conjecture: New Results from the Beginning of the 21st Century*. Frontiers in Mathematics. Birkhäuser, 2021. ISBN: 978-3-030-60533-9. DOI: https://doi.org/10.1007/978-3-030-60535-3.